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Exact analysis of the Peano basin

A Flammini and F Colaiori

Istituto Nazionale di Fisica della Materia (INFM), International School for Advanced Studies (SISSA), via Beirut 2-4, 34014 Trieste, Italy

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Abstract. The Peano basin is analysed as a deterministic model for river networks. The fundamental distributions characterizing the real basins morphology can be explicitly calculated. The recently proposed finite size-scaling ansatz is tested apart from oscillatory amplitudes typical of deterministic fractals.

1. Introduction

In recent years a variety of approaches towards the statistical characterization of river networks have been proposed, in the attempt to describe a basin's morphology [1–6].

In real drainage networks, rainfall collected by the basin flows downhill through channels, which, by means of erosion, self-organizes in a spatial tree-like structure. Such networks are known to exhibit power-law behaviour typical of fractal structures [7–9] in drainage sub-basin areas and mainstream-lengths distributions.

A river basin can be described giving a scalar field of elevations and defining drainage directions by steepest descents. Thus in a simple lattice picture a river network is represented by an oriented spanning tree over a two-dimensional lattice. To each site i one can associate a local injection of mass (the average annual rainfall at site i) that can be taken equal to 1. Then the flow A_i at site i, or equivalently the drained area at site i can be defined as the sum of the injection over all points upstream with respect to i (we say that site j is upstream with respect to site i if drainage directions go from j to i).

Variables A_i are thus related by:

$$A_i = \sum_j w_{i,j} A_j + r_i \tag{1}$$

where $w_{i,j}$ is 1 if site j is a nearest neighbour of i upstream with respect to site i and 0 otherwise. In natural basins these areas can be investigated through experimental analysis of digital elevation maps (DEM's) [3].

Another relevant quantity in a basin's morphology is the upstream length relative to a site, defined as the length of the stream obtained starting from the site and moving in the upstream direction towards the nearest neighbour with biggest area A (the one leading to the outlet excluded), since a source, i.e. a site with no incoming links, is reached. If two or more equal areas are encountered, one is randomly selected.

Recently a simple finite size-scaling ansatz has been proposed [10] leading to natural explanation of scaling properties. For a lattice of given linear size L call $p(A, L)$ the probability density distribution of cumulated areas A and $\pi(l, L)$ the probability density distribution of the upstream lengths l, i.e. the fraction of sites with, respectively, area A or

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stream length l. Consider also the integrated distributions $P(A, L)$, fraction of sites with cumulated area bigger then A and $\Pi(l, L)$, and fraction of sites with upstream length bigger than l. In this notation the finite size-scaling ansatz reads:

$$p(A, L) = A^{-\tau} f\left(\frac{A}{A_C}\right) \tag{2}$$

$$\pi(l, L) = l^{-\psi} g\left(\frac{l}{l_C}\right) \tag{3}$$

where $f(x)$ and $g(x)$ are scaling functions taking into account the finite size effect and a_C and l_C are characteristic area and length. Functions f and g are supposed to have the following properties: when $x \to \infty$ they go to zero sufficiently fast to ensure normalization; when $x \to 0$ they tend to a constant, to give power-law behaviour in the large size limit.

The characteristic area and length are supposed to scale respectively:

$$A_C \sim L^{\varphi} \tag{4}$$

$$l_C \sim L^{\delta}. \tag{5}$$

For the same quantities, integrated probability distributions can be analogously written:

$$P(A, L) = A^{1-\tau} F\left(\frac{A}{L^{\varphi}}\right) \tag{6}$$

$$\Pi(l, L) = l^{1-\psi} G\left(\frac{l}{L^{\delta}}\right) \tag{7}$$

which follow from (2) and (3) with

$$F(x) = x^{\tau-1} \int_x^{+\infty} \mathrm{d}y\, y^{-\tau} f(y) \tag{8}$$

$$G(x) = x^{\psi-1} \int_x^{+\infty} \mathrm{d}y\, y^{-\psi} g(y). \tag{9}$$

In the next section we will work out exactly the distributions with an explicit calculation in the case of the Peano basin. The scaling ansatz is found to be exact in this case and scaling exponents are computed. Moreover it will be shown how scaling exponents τ and ψ can be deduced by renormalization-group arguments. Numerical analysis of above distributions show the presence of an oscillatory term superimposed to the power laws that will find natural explanation in terms of renormalization-group analysis.

2. The Peano basin

The Peano basin [7, 11] is a deterministic space-filling fractal with a tree-like structure with some resemblance with that of real rivers, and is obtained as follows. At step 0 it is an oriented link. Step 1 is obtained by replacing such link with four new links: two resulting from the subdivision in half of the old link and preserving its orientation, the other two having a common extreme in the middle point of the old link and both oriented towards it (see figure 1). The basin at each successive step, is obtained by iterating the construction, i.e. replacing each link with four new oriented links in the same way. After T steps the fractal has $N_T = 4^T$ points (excluding the outlet) and it can be mapped onto a square lattice of size $L = 2^T$ with bonds connecting first and second neighbours to form a spanning tree.

We can associate to each site i of a T step Peano basin an area $A_i(T)$ defined as in (1). Let $\mathcal{V}_T$ denote the set of distinct values assumed by the variables $\{A_j\}$ at step T. This can

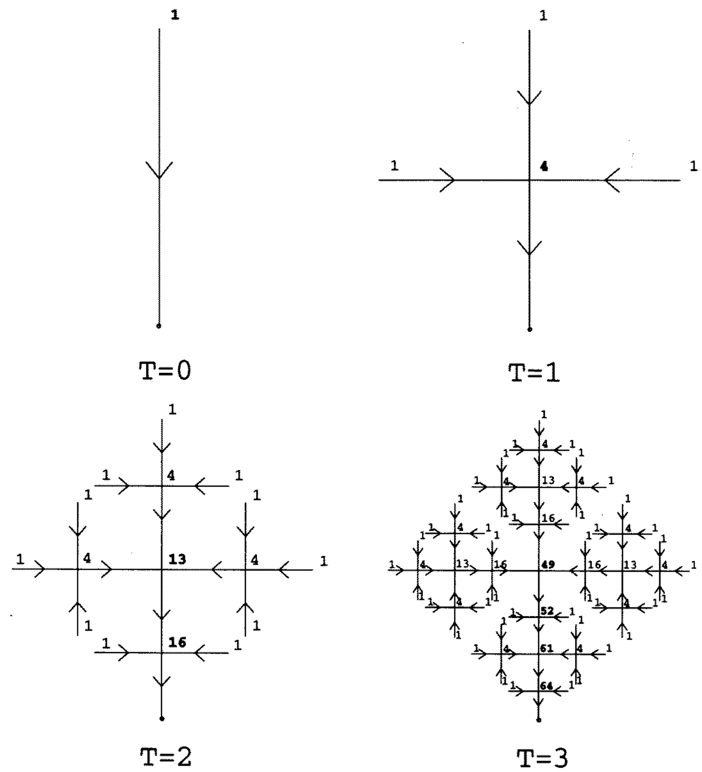

Figure 1. The Peano basin at iteration step $T = 0$, $T = 1$, $T = 2$, $T = 3$, with the cumulated area displayed.

be easily checked by induction, $\mathcal{V}_T$ contains $\mathcal{V}_{T-1}$ and 2^{T-1} new distinct values, appearing for the first time. Thus $\mathcal{V}_T$ is contained in all 2^T distinct numbers. Let us call $\mathcal{A} \doteq \bigcup_{T=0}^{\infty} \mathcal{V}_T$ and a_n the increasing sequence of numbers in $\mathcal{A}$ (the distinct values assumed by variables A_j iterating the construction). For such sequence, we found the following rule:

$$a_n = 3\left(\sum_k c_k(n)4^k\right) + 1 \qquad n = 0, 1, \ldots \tag{10}$$

where the $c_k(n)$ are the coefficients of the binary expansion of n:

$$n = \sum_k c_k(n)2^k. \tag{11}$$

Let M_n^T be the number of sites i at step T whose area A_i assume a given admissible value a_n. From the construction shown in figure 2 one can deduce that the following recursive relation holds:

$$\begin{cases} M_n^T = 4M_n^{T-1} - 1 & T > t(a_n) \\ M_n^T = 1 & T = t(a_n) \\ M_n^T = 0 & T < t(a_n) \end{cases} \tag{12}$$

where $t(a_n)$ denotes the a_n ‘borning time’, i.e. the first step in which an area with value a_n

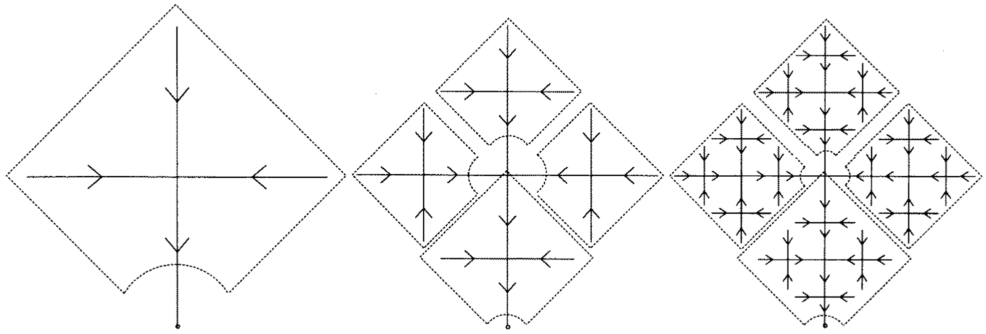

Figure 2. With this construction the recursive relation (12) can be easily understood. The Peano basin at time step $T+1$ is shown in terms of the one at time step T.

appears. This is easily seen to be given by

$$t(a_n) = \begin{cases} 0 & n = 0 \\ 1 + [\log_2(n)] & n > 0 \end{cases} \tag{13}$$

where $[\,\cdot\,]$ is the integer part.

Solving (12) one gets

$$M_n^T = \begin{cases} 0 & T < t(a_n) \\ \frac{2}{3}4^{T-t(a_n)} + \frac{1}{3} & T \geqslant t(a_n) \end{cases} \tag{14}$$

and thus all the a_n's 'born' at the same time step have the same probability

$$p_T(a_n) \doteq p(a_n, L = 2^T) = M_n^T / N_T. \tag{15}$$

Then the integrated distribution of areas $P(A_i > a_n, L = 2^T)$ assume a very simple expression for the a_n of the form 4^t (one can easily check from (10) that $a_{2^t-1} = 4^t$), and is given by (6)

$$P(A_i > a = 4^t, L = 2^T) = a^{1-\tau} F\left(\frac{a}{L^\varphi}\right) \tag{16}$$

with

$$\tau = \tfrac{3}{2} \qquad \varphi = 2 \tag{17}$$

and

$$F(x) = \tfrac{1}{3}(1-x) \qquad 0 < x < 1 \tag{18}$$

and $F(x) = 0$ when $x > 1$.

Equation (16) can be obtained observing that $P(A_i > a = 4^t, L = 2^T) = \sum_{n=2^t}^{2^T} p_T(a_n)$ depends on n only through $t(a_n)$, allowing the replacement of the sum over n with a sum over the steps s; moreover, for each step $s > 0$ there are 2^{s-1} areas with the same $t(a_n) = s$, thus

$$\begin{aligned} P(A_i > a = 4^t, L = 2^T) &= \sum_{s=t+1}^{T} (\tfrac{2}{3}4^{(T-s)} + \tfrac{1}{3})\frac{2^{s-1}}{4^T} \\ &= \tfrac{1}{3}2^{-t}(1 - 2^{2(t-T)}) = \tfrac{1}{3}a^{-\frac{1}{2}}\left(1 - \frac{a}{L^2}\right) \end{aligned} \tag{19}$$

which yields (16)–(18). Similarly, choosing l of the form $l = 2^t$ and observing that at step T the sites with upstream length bigger or equal to 2^t are the ones in which the cumulative area exceeds 4^t, we easily find:

$$\Pi(l \geqslant 2^t, L = 2^T) = l^{1-\psi} G\left(\frac{l}{L^\delta}\right) \tag{20}$$

which is of the form (7) with

$$\psi = 2 \qquad \delta = 1 \tag{21}$$

and

$$G(x) = \tfrac{1}{3}(1 - x^2). \tag{22}$$

Scaling exponents for the Peano basin can also be obtained by a renormalization-group argument. Let us consider for instance the scaling of cumulated areas. The self-similar structure of the Peano basin suggests a natural 'decimation' procedure [12]. The idea is the following: consider the equations relating areas at time-step T; then, eliminate variables related to sites that are not present at time step $T - 1$. This leads to a reduced equation describing the same physics on a tree scaled down by a factor of 2.

For the sake of simplicity let us consider the Peano basin at the second step of iteration. In figure 3 $A_n^{(2)}$ denote the variables related to sites that are present at step $T = 1$ and $B_n^{(2)}$ denote the ones that will be eliminated by decimation. The upper label refers to the step. In what follows it will be useful to write the equations in terms of $\tilde{A}_n^{(T)} = A_n^{(T)} - 1$ and $\tilde{B}_n^{(T)} = B_n^{(T)} - 1$. The areas at step $T = 2$ are related to each other by:

$$\begin{aligned} \tilde{A}_1^{(2)} &= 3\tilde{B}_1^{(2)} + 3 \\ \tilde{B}_1^{(2)} &= \tilde{A}_0^{(2)} + 2\tilde{B}_0^{(2)} + 3 \\ \tilde{B}_0^{(2)} &= 0. \end{aligned} \tag{23}$$

Elimination of the $\tilde{B}_n^{(2)}$ leads to

$$\tilde{A}_1^{(2)} = 3\tilde{A}_0^{(2)} + 12. \tag{24}$$

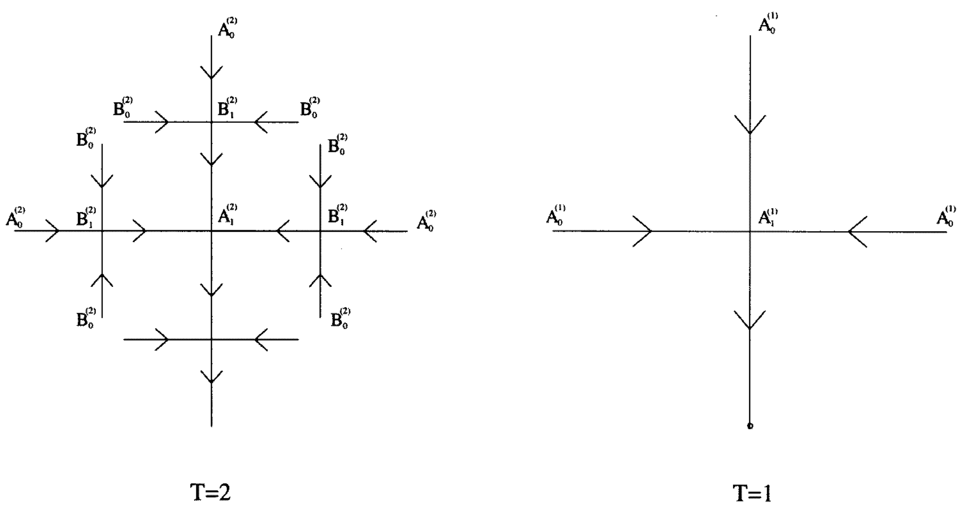

Figure 3. The renormalization group argument for the Peano basin. B-sites die under decimation.

At time step $T = 1$ the relation between areas is straightforward:

$$\tilde{A}_1^{(1)} = 3\tilde{A}_0^{(1)} + 3. \tag{25}$$

Equations (24) and (25) are the same if

$$\tilde{A}_n^{(T+1)} = 4\tilde{A}_n^{(T)} \tag{26}$$

i.e.

$$(A_n^{(T+1)} - 1) = 4(A_n^{(T)} - 1). \tag{27}$$

Denoting with $n^{(T+1)}(a)$ the number of sites with area greater then a at step $T + 1$, one can easily observe that the number of decimated sites with $A > a$ is half of the total number of sites with $A > a$

$$n^{(T+1)}(a) = 2n^{(T)}(a/4) \tag{28}$$

thus, being the total number of sites at step T, $N_T = 4^T$, it follows for the integrated probability $P(A_n^{(T+1)} > a) = \frac{n^{(T)}(a)}{N_T}$:

$$P(A_n^{(T+1)} > a) = bP(A_n^{(T)} > \Lambda a) \tag{29}$$

with

$$\Lambda = \tfrac{1}{4} \qquad \text{and} \qquad b = \frac{n^{(T+1)}(a)/4^{T+1}}{n^{(T)}(a)/4^T} = \frac{2/4^{T+1}}{1/4^T} = \frac{1}{2}. \tag{30}$$

Equation (29) rewritten in terms of $P(a) = \tilde{P}(\log a)$ as

$$\tilde{P}(x) = b\tilde{P}(x + \lambda) \tag{31}$$

where $\lambda = \log \Lambda$. The general solution is $\tilde{P}(x) = \exp[(1 - \tau)x]w(x)$, where $w(x)$ is a periodic function of period $|\lambda|$ and $\exp[(1 - \tau)\lambda] = \Lambda^{1-\tau} = b^{-1}$. Using (30) $\tau = \frac{3}{2}$; then the solution of (29) is

$$P(a) = a^{1-\tau}w(\log a). \tag{32}$$

Thus an oscillatory term in $\log a$ superimposed to the power law $a^{(1-\tau)}$ might exist and in effect it does. The same argument can be repeated for the distribution of mainstream lengths, recovering the ψ exponent derived in (21).

The possible existence of oscillatory terms was first noted by Nauenberg [13] and has been discussed by Niemeijer and van Leeuwen in [14]. They report an argument due to Nelson claiming that oscillatory terms are unlikely to appear because of the fact that two renormalization transformations with opportunely chosen scales factors would lead to incommensurate periods λ_1, λ_2 and thus to a constant $w(x)$. In our case this argument does not apply since we can have only $b = 2^k$, i.e. $\lambda = -\frac{1}{\tau-1}k\log 2$.

In figure 4 the log–log plot of $P(a)$ versus a, as obtained using (15) for the probability density, is shown for a system of linear size $L = 2^{14}$. The broken line is the power law $a^{(1-\tau)}$ times the scaling function $F(a/L^2)$ of (18). In order to highlight the periodicity of the function $w(x)$ of (32) in figure 5 we draw a log–log plot of $P(a)a^{(\tau-1)} = w(\log a)F(a/L^2)$ versus a. The broken line is $F(a/L^2)$. The periodicity is $|\lambda| = \log 4$. The same plots have been done for the mainstream length distribution and are shown in figures 6 and 7.

Note that the same renormalization-group argument can be done in generic dimension d giving $\tau = 1 + \ln d/\ln(2d)$ and $\psi = 1 + \ln d/[2\ln(2d)]$.

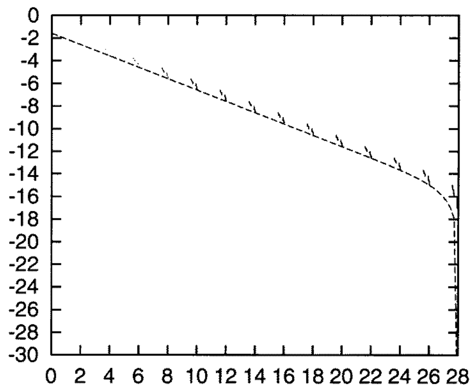

Figure 4. Log–log plot of the cumulated areas distribution $P(a)$ versus a for a system of linear size $L = 2^{14}$. The broken line is $a^{(1-\tau)}F(a/L^2)$.

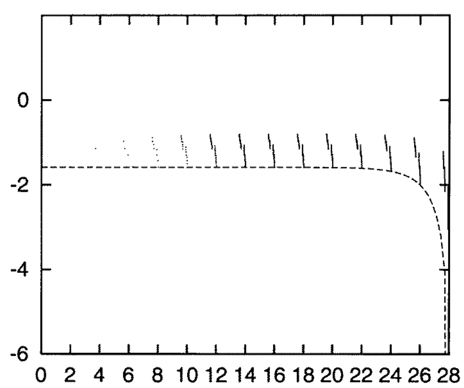

Figure 5. Log–log plot of $P(a)a^{(\tau-1)}$ versus a. The broken line is $F(a/L^2)$ in (18). The periodicity is log 4.

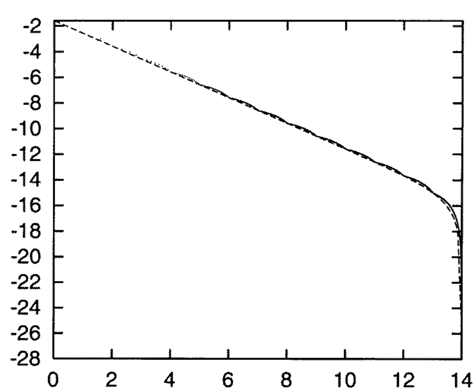

Figure 6. The same as in figure 4 for the mainstream-lengths distribution.

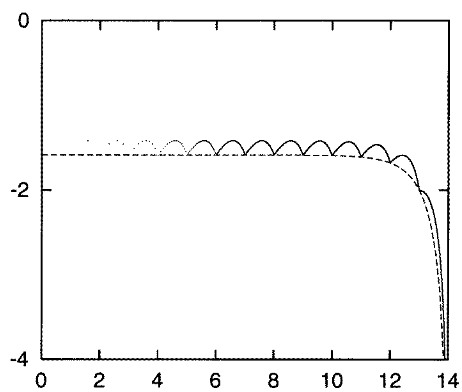

Figure 7. The same as in figure 5 for the mainstream-lengths distribution. Note that in this case the period is $|\lambda| = \frac{1}{2}$.

3. Conclusions

In this paper we analysed in detail the Peano basin. The scaling behaviour has been worked out exactly. Similar lattice models for rivers have recently been the object of extensive studies [15] in which various universality classes have been identified. In particular the mean-field theory has been solved through a mapping on the Takayasu random aggregation model [16]. Note that scaling exponents for the Peano basin in $d = 2$ result are the same at this mean field.

A remark needs to be made regarding the generalization in dimension d. One should expect that in the limiting $d \to \infty$ the exponents tend to the mean-field ones. Actually this is not the case, in fact one gets $\tau_{d\to\infty} = 2$, $\psi_{d\to\infty} = \frac{3}{2}$. This is not surprising, and is due to the fact that in the Takayasu model the tree is space filling only up to dimension $d = 2$ [16] while in the Peano basin we impose that by construction in all d.

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